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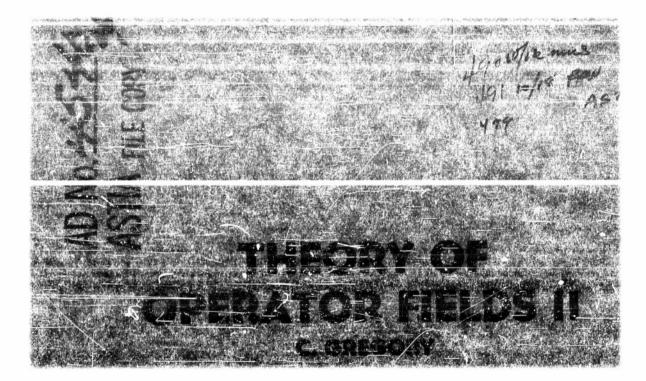APPROVED FOR PUBLIC RELEASE;
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THEORY OF

OPERATOR FIELDS II

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THEORY OF OPERATOR FIELDS II

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CONTENTS

ABSTRACT

One of the basic difficulties (aside from the mathematical) in
the theory of non-local fields has to do with physical interpretation.
In the case of "free fields" in so far as our formalism is concerned
there does not appear to be much difference between local and non-
local fields. It was for this reason that we embarked on a program
involving the study of non-local fields with interactions chosen in
such manner as to vanish in the limit of the local field approximation,
if indeed the fields in nature are non-local to begin with. Our con-
siderations are restricted to the situation which maintains for the
case of the so-called non-local electromagnetic field as a convenient
example since the properties of the ordinary electromagnetic field are
well known. Although the form of the interaction that we have chosen
is somewhat artificial, it satisfies the requirement of vanishing in the
local field case and brings us in contact with the type of problem that
has to be dealt with. In particular the interaction is chosen to be a
local external inhomogeneous electromagnetic field and is introduced
in a manner similar to that currently in vogue in the Quantum Theory
Fields.

It is shown that our problem can be reduced to the solution of
the problem occurring in the usual theory for a "massless" vector
meson in an equivalent external electromagnetic field. This external

electromagnetic field occurs as the difference of the local external

field evaluated at space-time points separated by the C-numbers

q_μ which occur in the expansion of a non-local field. Thus a non-

local "neutral" particle characterized additionally by the numbers

q_μ would take on the characteristic of a "charged" particle under

influence of the quivalent external electromagnetic field. On the basis

of this finding it would be possible to take over all of the machinery of

current theory and at the same time inherit their shortcomings.

Although no numerical calculations are made, this finding increases

our insight regarding physical significance, in so far as our work is

concerned, of the C-numbers q_μ appearing in non-local field

theory. This result is to be compared with the significance attributed

to the q_μ for the case of a constant external E, H field.[1]

Second quantization of the free non-local field is effected in a

manner different from that effected for a scalar field reported else-

where[2] with substantially the same results. In particular, here,

upon anticipating second quantization, the fields are subject to a

unitary transformation which in a certain sense localizes the field.

Again as in the case for a scalar field[2] the local and non-local

electromagnetic fields admit of a net number of quanta operator with

the aid of which the energy and momentum and other operators for the

field in particular the mass operator may be expressed. The energy

is positive definite while the momentum and net number of quanta operator are not.

Some consideration is given to the problem of the generalized Schroedinger equation for unitary transformations in order to yield independently of the commutation relationships postulated for the fields the result that if the field equations are satisfied the unitary transformation considered as a functional of a space like surface is independent of the latter. This is effected for fields obtained via the usual Lagrangian methods as well as via operational methods for local fields. In the latter an essential difference occurs due to the appearance of the net number of quanta operator in the functional equation for the elements of the unitary transformation. This may have a bearing on the concept of statistical equilibrium in that the probability for transitions from one state to another (under suitable restrictions) is found to decrease as the net number of quanta involved increases.

The results of our investigation amongst other things indicate that the operational methods yield novel results even for local fields as exemplified, for example, by the non-vanishing of the net number of quanta operator even in the limit of operator local fields. This operator vanishes in the limit of C- number fields. It can be shown that if we admit complex fields (Spinor fields satisfying first order equations) the four-current quantum density obtained via conventional

methods is identical aside from numerical factors from the one obtained via our operational methods. Such a current would vanish if the fields are real according to conventional techniques. Our operational method indicates that a four-current vector exists whether the fields are complex or not. This raises the question among many as to the possibility of describing all fields by means of real (hermitian) entities since heretofore the complex fields were introduced to assure existence of a four-current vector. It is believed that this question as well as others warrants investigation even within the domain of local operator fields.

I. INTRODUCTION

1. Problem of Non-local Photon in Interaction with General Local E, H Field.

In an earlier report[3] consideration was given to the problem of the interaction between a constant E, H field and a non-local photon from a semi-classical point of view. We have left open the question of quantization of such fields. In the case of a constant E, H field the operator field equations were not too difficult to solve. For a general E, H field the equations are such as to necessitate our restricting the investigation to a study of the possibility of reducing the problem to that which confronts us in the ordinary theory. We will find that this can be done in a mathematical sense. However, the reduction of the problem will not imply that no physical differences occur between the theories. The structure of the reduced equations are such as to help us make some physical interpretations (including quantization) using our classical theory as a guide.

2. Some Identities.

Let us consider the commutator expression

$$\zeta_\mu \equiv i \left[p_\mu - V_\mu, U \right], \tag{1.1}$$

where V_μ is a local function:

$$[P_\mu, V] = 0, \tag{1.2}$$

and P_μ is one of the displacement operators. U on the other hand is a non-local function. Moreover, we recall[3] the following fundamental commutation relationships between the displacement operators P_μ and the space-time operators x^ν

$$[P_\mu, x^\nu] = -i \delta_\mu^\nu \tag{1.3}$$

for a system of units corresponding to the choice $\hbar = c = 1$.
Now V_μ and U may be expanded in terms of the basic functions $e^{ik_\mu x^\mu}$ and the non-local basic functions $e^{iq^\mu P_\mu} e^{ik_\mu x^\mu}$ respectively as

$$V_\mu = \sum_k v_\mu(k) e^{ik_\mu x^\mu}, \tag{1.4}$$

$$U = \sum_{k,q} u(q,k) e^{iq^\mu P_\mu} e^{ik_\mu x^\mu} \tag{1.5}$$

(1.4) is merely the ordinary Fourier expansion of a local function while (1.3) corresponds to the non-local Fourier expansion of a non-local function.[4] It is to be noted that $\sum_k$ and $\sum_{k,q}$ are to be

interpreted as 4-fold and 8-fold integrals over the $k-$ space and k, q space respectively unless otherwise specified. Upon introducing (1.4) and (1.5) into (1.1) we find that (1.1) may be written as

$$C_\mu = i \sum_{k',q} \left[k'_\mu \, u(q,k') - \sum_{k''} v^-_\mu (k'-k'')(e^{-iq^\mu (k'_\mu - k''_\mu)} - 1) \times \right.$$
$$\left. u(q,k'') \right] e^{iq^\mu p_\mu} e^{ik'_\mu x^\mu}, \qquad (1.6)$$

upon noting that

$$e^{ik_\mu x^\mu} e^{iq^\mu p_\mu} = e^{-iq^\mu k_\mu} e^{iq^\mu p_\mu} e^{ik_\mu x^\mu}, \qquad (1.7)$$

and redefining our summation variables.

The coefficients of our basic functions $e^{iq^\mu p_\mu} e^{ik'_\mu x^\mu}$ occurring in (1.6) bear a striking resemblance to the operations involving matrices and vectors in quantum mechanics. For if we define (in a representation with k diagonal)

$$\langle k' | f(k) | k'' \rangle = f(k') \, \delta(k'-k''), \qquad (1.8)$$

$$\langle k' | v_\mu(q) | k'' \rangle = v_\mu (k'-k'')(e^{-iq^\mu (k_\mu - k''_\mu)} - 1), \qquad (1.9)$$

then (1.6) may be written as

$$C_\mu = i \sum_{q, k', k''} \langle k' | k_\mu - v_\mu(q) | k'' \rangle u(q, k'') e^{i q^\mu p_\mu} e^{i k_\mu x^\mu}. \quad (1.10)$$

Now if we undertake to expand

$$C_{\nu\mu} \equiv i \left[p_\nu - V_\nu, C_\mu \right], \quad (1.11)$$

where V_ν is another local function with expansion

$$V_\nu = \sum_k v_\nu(k) e^{i k_\mu x^\mu}, \quad (1.12)$$

we will discover after using some elementary laws of matrix multiplication that

$$C_{\nu\mu} = (i)^2 \left[[p_\nu - V_\nu], [p_\mu - V_\mu, U] \right]$$

$$= -\sum_{q, k', k''} \langle k' | (k_\nu - v_\nu)(k_\mu - v_\mu) | k'' \rangle u(q, k'') \times$$

$$e^{i q^\mu p_\mu} e^{i k'_\mu x^\mu} \quad (1.13)$$

These results are readily generalized to commutator expressions of higher order.

It may be more convenient to work in a representation with the operators ξ^μ diagonal which we define by the equations

$$[\xi^{\mu}, \xi^{\nu}] = 0$$

$$[k_{\mu}, \xi^{\nu}] = -i \delta_{\mu}^{\nu}$$

$$(1.14)$$

By the analogy which exists between (1.14) and the commutator equations involving the space-time operators χ^{μ} and the displacement operators p_{μ} we can with the aid of our transformation functions

$$\langle \xi' | k' \rangle = (4\pi^2)^{-1} e^{ik'_{\mu} \xi'^{\mu}} = \langle k' | \xi' \rangle^{*} \qquad (1.15)$$

rewrite (1.10) in terms of the ξ's . Let us obtain the structure of $\langle k' | v_{\mu} | k'' \rangle$ given by (1.9). Since from (1.4) V_{μ} is local we may write

$$V_{\mu}(\chi) = \sum_{k} e^{ik\chi} V_{\mu}(i\frac{\partial}{\partial k}) \delta(k) \qquad (1.16)$$

Consequently, from (1.9) we have

$$\langle k' | v_{\mu}(q) | k'' \rangle = (e^{-iq^{\mu}(k'_{\mu} - k''_{\mu})} - 1) \times$$

$$V_{\mu}(i\frac{\partial}{\partial k'}) \delta(k' - k''), \qquad (1.17)$$

(1.17) is readily recognized to be the matrix element of
$V_\mu(\xi - q) - V_\mu(\xi)$. Hence we may write

$$\langle k' | \bar{v}_\mu(q) | k'' \rangle = \langle k' | V_\mu(\xi - q) - V_\mu(\xi) | k'' \rangle, \qquad (1.18)$$

which in view of (1.14) enables us to write in a representation with
ξ diagonal

$$\langle \xi' | v_\mu(q) | \xi'' \rangle = [V_\mu(\xi' - q) - V_\mu(\xi')] \delta(\xi' - \xi'') \qquad (1.19)$$

II. EQUATIONS FOR NON-LOCAL ELECTROMAGNETIC FIELD IN INTERACTION WITH A LOCAL ELECTROMAGNETIC FIELD

2. Field Equations.

From previous work[3] the field equations for a non-local Electromagnetic Field in interaction with a local external electromagnetic field are

$$ i\,[\,P_\mu, F^{\mu\nu}\,] = 0, \tag{2.1} $$

where

$$ F_{\mu\nu} \equiv i\,[\,P_\mu, A_\nu\,] - i\,[\,P_\nu, A_\mu\,], \tag{2.2} $$

with P_μ defined in terms of the displacement operator p_μ and the external local vector potential $A_\mu^{(e)}$ as

$$ P_\mu = p_\mu - g\,A_\mu^{(e)}, \tag{2.3} $$

where g is the coupling constant and where we have taken $\hbar = c = 1$. In general we cannot add in addition to (2.1) the equations

$$ [\,P_\mu, F_{\nu\sigma}\,] + [\,P_\nu, F_{\sigma\mu}\,] + [\,P_\sigma, F_{\mu\nu}\,] = 0 $$

unless $[P_\mu, P_v] = $ const., which latter occurs only if the external E, H field is constant. This case has already been considered. [1]

Now

$$i^{-1}[P_\sigma, F_{\mu v}] = [P_\sigma, P_\mu, A_v] - [P_\sigma, P_v, A_\mu]$$

$$i^{-1}[P_\mu, F_{v\sigma}] = [P_\mu, P_v, A_\sigma] - [P_\mu, P_\sigma, A_v]$$

$$i^{-1}[P_v, F_{\sigma\mu}] = [P_v, P_\sigma, A_\mu] - [P_v, P_\mu, A_\sigma]$$

$$(2.4)$$

upon using (2.2). But

$$[P_\sigma, P_\mu, A_v] - [P_\mu, P_\sigma, A_v] =$$

$$[P_\sigma, P_\mu, A_v] + [P_\sigma, A_v, P_\mu] + [A_v, P_\mu, P_\sigma] \qquad (2.5)$$

after using the Jacobi identity

$$[A, B, C] + [B, C, A] + [C, A, B] \equiv 0. \qquad (2.6)$$

Consequently, upon noting that the first two terms on the right side of (2.5) make no contribution we have upon adding the equations of (2.4)

$$i^{-1}[P_\sigma, F_{\mu\nu}] + i^{-1}[P_\mu, F_{\nu\sigma}] + i^{-1}[P_\nu, F_{\sigma\mu}] \equiv$$

$$[A_\nu, P_\mu, P_\sigma] + [A_\sigma, P_\nu, P_\mu] + [A_\mu, P_\sigma, P_\nu] =$$

$$-[A_\sigma, P_\mu, P_\nu] - [A_\mu, P_\nu, P_\sigma] - [A_\nu, P_\sigma, P_\mu]. \tag{2.7}$$

However,

$$[P_\mu, P_\nu] = -i^{-1} g \, F_{\mu\nu}^{(e)}, \tag{2.8}$$

where $F_{\mu\nu}^{(e)}$ is the local external field strength. In view of (2.8), (2.7) may be written as

$$[P_\sigma, F_{\mu\nu}] + [P_\mu, F_{\nu\sigma}] + [P_\nu, F_{\sigma\mu}] =$$

$$g[A_\sigma, F_{\mu\nu}^{(e)}] + g[A_\mu, F_{\nu\sigma}^{(e)}] + g[A_\nu, F_{\sigma\mu}^{(e)}]. \tag{2.9}$$

Equation (2.9) is the analogue for our case of the second set of Maxwell's equations which are identities as a consequence of the definition of the field strengths in terms of the vector potentials. Inspection of (2.1) and (2.9) shows that these equations are invariant if we add to the vector potentials A_μ the expressions $[P_\mu, \psi]$ where ψ

is a local function. For if we denote by $F'_{\mu\nu}$ the $F_{\mu\nu}$ obtained by augmenting each A_μ by $[p_\mu, \psi]$ we obtain

$$F'_{\mu\nu} = F_{\mu\nu} + [A^{(e)}_\mu, p_\nu, \psi] - [A^{(e)}_\nu, p_\mu, \psi], \qquad (2.10)$$

which reduces to

$$F'_{\mu\nu} = F_{\mu\nu} \qquad (2.11)$$

upon making the observation that if ψ is a local function so is $[p_\mu, \psi]$ and $[p_\nu, \psi]$ and then noting that the external vector potentials $A^{(e)}_\nu$ and $A^{(e)}_\mu$ are local functions.

If now we alter the external $A^{(e)}_\mu$ by adding $[p_\mu \psi]$ we observe that

$$F''_{\mu\nu} = F_{\mu\nu} - [A_\nu, p_\mu, \psi] + [A_\mu, p_\nu, \psi], \qquad (2.12)$$

where $F''_{\mu\nu}$ denotes the altered field strengths. Consequently, if we alter both the external and internal vector potentials by adding $[p_\mu, \psi]$, we find that

$$F'''_{\mu\nu} = F''_{\mu\nu} \neq F_{\mu\nu} \qquad (2.13)$$

where $F_{\mu\nu}'''$ corresponds to the altered $F_{\mu\nu}''$. Thus equations (2.1) and (2.9) are altered. We must conclude then that only if the internal vector potentials A_μ are subject to a gauge transformation will the Field equations be invariant unless the fields are local and/or $[p_\mu, \Psi]$ is constant.

4. Equations for Coefficients in the Expansion of the Non-local Vector Potentials A_μ:

If we introduce (2.2) in (2.1) we obtain

$$-[P_\mu, P^\mu, A^\nu] + [P_\mu, P^\nu, A^\mu] = 0. \qquad (2.14)$$

Consequently, if we identify V_μ of section 2 with $g A_\mu^{(e)}$ which appears in our definition of P_μ in (2.3) we obtain upon introducing the expansions

$$A^\mu \equiv \sum_{q,k} a^\mu(q,k) e^{i q^\mu p_\mu} e^{i k_\mu x^\mu}, \qquad (2.15)$$

$$A^{(e)\mu} \equiv \sum_k a^{(e)\mu}(k) e^{i k_\mu x^\mu}, \qquad (2.16)$$

into (2.14)

$$\sum_{q,k',k''} \left\{ \langle k' | (k_\mu - g A_\mu^{(e)}(\xi - q) + g A_\mu^{(e)}(\xi))(k^\mu - g A^{(a)\mu}(\xi - q) + g A^{(e)\mu}(\xi)) | k'' \rangle \times \right.$$

$$a^\nu(q, k'') - \langle k' | (k_\mu - g A_\mu^{(e)}(\xi - q) + g A_\mu^{(e)}(\xi))(k^\nu - g A^{(e)\nu}(\xi - q) + $$

$$\left. g A^{(e)\nu}(\xi)) | k'' \rangle a^\mu(q, k'') \right\} e^{i q^\mu p_\mu} e^{i k'_\mu x^\mu} \qquad (2.17)$$

Now we recall[3], [4] that if

$$U = \sum_{q,k} a(q,k) \, U_{qk},$$

where the U_{qk} are basic functions, then

$$a(q,k) = \mathrm{Tr} \, U \, \widetilde{U}_{qk},$$

so that if $U = O$, then $a(q,k) \neq O$. From this we must conclude since $e^{i q^\mu p_\mu} e^{i k'_\mu x^\mu}$ are proportional to the non-local Fourier basic functions that (2.17) is equivalent to

$$\sum_{k''} \left\{ \langle k' | (k_\mu - g A_\mu^{(e)}(\xi - q) + g A_\mu^{(e)}(\xi))(k^\mu - g A^{(e)\mu}(\xi - q) + \right.$$

$$g A^{(e)\mu}(\xi)) | k'' \rangle a^\nu(q, k'')$$

$$- \langle k' | (k_\mu - g A_\mu^{(e)}(\xi - q) + g A_\mu^{(e)}(\xi))(k^\nu - g A^{(e)\nu}(\xi - q) + $$

$$\left. g A^{(a)\nu}(\xi)) | k'' \rangle a^\mu(q, k'') \right\} \neq O. \qquad (2.18)$$

If we refer to the work of section 2 again we discover that (2.18) may
be given an operator version if we interpret

$$\xi_\mu = -\frac{1}{i}\frac{\partial}{\partial k^\mu},\tag{2.19}$$

so that (2.18) may be written as

$$\left(k_\mu - g A_\mu^{(e)}(\xi - q) + g A_\mu^{(e)}(\xi)\right)\left(k^\mu - g A^{(e)\mu}(\xi - q) + g A^{(e)\mu}(\xi)\right)a^\nu(q,k)$$

$$-\left(k_\mu - g A_\mu^{(e)}(\xi - q) + g A_\mu^{(e)}(\xi)\right)\left(k^\nu - g A^{(e)\nu}(\xi - q) + g A^{(a)\nu}(\xi)\right)a^\mu(q,k)=0.\tag{2.20}$$

Another form of (2.18) may be obtained by going to a representation
with the ξ's diagonal. This may be done with the aid of our trans-
formation functions (1.15) as in quantum mechanics. For if we multi-
ply both sides of (2.18) by the transformation function $\langle \xi' \mid k' \rangle$
and integrate using

$$\int \mid k' \rangle \, d^4 k' \langle k' \mid = 1,$$

and then expressing $\mid k'' \rangle$ as

$$\mid k'' \rangle = \int \mid \xi'' \rangle \langle \xi'' \mid k'' \rangle d^4 \xi'',$$

we obtain the operator version in this representation (ξ diagonal) to be

$$\left(k_\mu - g A_\mu^{(e)}(\xi - q) + g A_\mu^{(e)}(\xi)\right)\left(k^\mu - g A^{(e)\mu}(\xi - q) + g A^{(e)\mu}(\xi)\right)\psi^\nu(q,\xi)$$

$$- \left(k_\mu - g A_\mu^{(e)}(\xi - q) + g A_\mu^{(e)}(\xi)\right)\left(k^\nu - g A^{(e)\nu}(\xi - q) + g A^{(e)\nu}(\xi)\right)\psi^\mu(q,\xi) = 0. \quad (2.21)$$

where

$$\psi^\mu(q,\xi) \equiv \frac{1}{4\pi^2}\int e^{i\xi k}\, a^\mu(q,k)\, d^4k. \tag{2.22}$$

We have now succeeded in reducing the solution of the original operator field equations (2.14) to the solution of operator equations which customarily appear in quantum mechanics, namely (2.20) or (2.21). The $a^\mu(q,k)$ appearing in (2.20) and the $\psi^\mu(q,\xi)$ appearing in (2.21) may be looked upon as vector wave functions, the former being in a representation with ξ diagonal and the latter in a representation with k diagonal, with ξ and k satisfying the commutation relationships (1.14) and further where the a's and ψ's are related through (2.22).

5. Equivalent Electromagnetic Field.

Further inspection of (2.21) indicates the striking similarity

which exists between it and the classic field equations for a massless

vector meson in interaction with an external electromagnetic field

$$A_\mu^{(e)}(\xi - q) - A_\mu^{(e)}(\xi)$$ which latter would vanish in the case

of local fields

$$q^\mu = 0.$$

It would be appropriate to identify

$$A_\mu'^{(e)} \equiv A_\mu^{(e)}(\xi - q) - A_\mu^{(e)}(\xi) \tag{2.23}$$

as the equivalent external electromagnetic field that affects a non-

local photon from a purely structural point of view. Thus a non-local

"neutral" particle characterized by the numbers q_μ in a local electro-

magnetic field $A_\mu^{(e)}(x)$ would "see" or seem to be affected in the

classical sense by the equivalent electromagnetic field given by the

right hand side of (2.23). Thus if we impute to the $C-$ numbers ξ

appearing in (2.23) a rough equivalence to the numbers used to describe

the position of a point particle in classical mechanics, we may come

to some qualitative conclusions regarding the behavior of a non-local

"neutral" particle in a local electromagnetic field by invoking the

machinery of classical mechanics. As an example we may consider

the effect of a local coulomb field on a non-local massive particle.

From (2.23) we would conclude that the "trajectory" would be equiva-
lent to that which would exist for a charged particle in the presence
of two equal but oppositely charged particles displaced by a distance
$\left| \underset{\sim}{q} \right|$. From this point of view we have discovered another way to
give significance to the $C-$ numbers q appearing in our non-local
plane waves. For in the treatment of the problem of a non-local
photon in interaction with a constant E, H field[1] it was found that
the $C-$ numbers q appeared in such a fashion in the expressions
for the temporal component of k_μ as to make it possible to ascribe
to the non-local photon a magnetic and electric dipole moment.

It would seem possible to carry the analogy further and utilize
all of the formalisms at our disposal to carry on the program of
second quantization of the ψ_μ fields appearing in (2.21). Such a
procedure would imply the imputation of the same physical signifi-
cance to the ψ_μ as was accorded to the A_μ from the local point
of view. However, it must be recalled that we are restricting our
investigation to the case of a possible interaction of a local E, M
field with a non-local field and the conclusions and identifications
that have been suggested here may not be valid for the problem of
the interaction of two or more non-local fields.

The connection existing between A^μ and ψ^μ may be
readily obtained from the following considerations. From (2.22) we
have

$$\psi^\mu(q,\xi) = (4\pi^2)^{-1} \int e^{i \xi^\mu k_\mu} a^\mu(q,k) d^4 k \qquad (2.24)$$

But

$$A^\mu = (4\pi^2)^{-1} \iint a^\mu(q,k) e^{i q^\mu p_\mu} e^{i k_\mu x^\mu} d^4 q \, d^4 k, \qquad (2.25)$$

from (2.15) upon using the factor $(4\pi^2)^{-1}$ so that now the co-efficients of $a^\mu(q,k)$ in (2.25) involve our normalized non-local plane waves or basic functions. After introducing (2.24) in (2.25) we obtain

$$A^\mu = (16\pi^4)^{-1} \iiint e^{i q^\mu p_\mu} \psi^\mu(q,\xi) e^{i k_\mu(x^\mu - \xi^\mu)} d^4 k \, d^4 q \, d^4 \xi. \qquad (2.26)$$

However,

$$(16\pi^4)^{-1} \int \psi^\mu(q,\xi) e^{i k_\mu(x^\mu - \xi^\mu)} d^4 k \, d^4 \xi = \psi(q,x). \qquad (2.27)$$

To show this it is merely necessary to take the matrix elements of the left hand side of (2.27) in a representation with x diagonal to discover that it is equal to $\psi(q,x') \, \delta(x' - x'')$. Consequently, (2.26) becomes

$$A^\mu = \int e^{i q^\mu p_\mu} \psi^\mu(q,x) d^4 q. \qquad (2.28)$$

As a summary we may write the following equivalent expressions for a non-local function A^μ

$$
\begin{aligned}
A^\mu &= (4\pi^2)^{-1} \iint a^\mu(q,k)\, e^{iq^\mu p_\mu}\, e^{ik_\mu x^\mu}\, d^4q\, d^4k, \\
&= \int e^{iq^\mu p_\mu}\, \psi^\mu(q,x)\, d^4q,
\end{aligned}
\qquad (2.29)
$$

where

$$
\psi^\mu(q,\xi) = (4\pi^2)^{-1} \int e^{i\xi^\mu k_\mu}\, a^\mu(q,k)\, d^4k
$$

(2.29) is quite general.

6. Differential Equation for Matrix Elements.

In view of some previous work[3] we may express the non-local function A^μ by means of the equation

$$
A^\mu = \int e^{iq^\mu p_\mu} \langle x - q \,|\, A^\mu \,|\, x \rangle\, d^4q, \qquad (2.30)
$$

so that upon comparing with (2.28) we conclude that

$$
\psi^\mu(q,x) = \langle x - q \,|\, A^\mu \,|\, x \rangle. \qquad (2.31)
$$

(2.31) states that $\psi'^\mu(q, x)$ is obtained from the matrix elements $\langle x' | A^\mu | x'' \rangle$ by replacing x' by $x - q$ and x'' by x. (2.30) can also be written as

$$A^\mu = \int \langle x | A^\mu | x + q \rangle \, e^{i q^\mu P_\mu} \, d^4 q, \qquad (2.32)$$

upon noticing that

$$e^{i q^\mu P_\mu} \, f(x, p) \, e^{-i q^\mu P_\mu} = f(x + q, p). \qquad (2.33)$$

(2.30) and (2.31) suggest that we could have obtained (2.21) or its equivalent by making the substitution (2.30) or (2.32) directly into (2.1) and using the statement that if

$$\int \langle x | U | x + q \rangle \, e^{i q^\mu P_\mu} \, d^4 q = 0, \qquad (2.34)$$

then

$$\langle x | U | x + q \rangle = 0. \qquad (2.34a)$$

This statement may be verified in the following fashion. If we take the matrix elements of (2.34) in a representation with x diagonal we obtain upon noting that

$$\langle x' | e^{i q^\mu p_\mu} | x'' \rangle = \delta(x' - x'' + q),$$

$$\langle x' | \int \langle x | U | x+q \rangle e^{i q^\mu p_\mu} | x+q \rangle e^{i q^\mu p_\mu} d^a q | x'' \rangle = \langle x' | U | x'' \rangle.$$

Consequently, if (2.34) is satisfied we must have $\langle x' | U | x'' \rangle = 0$. Hence if we replace x' by x and x'' by $x+q$ we must conclude that (2.34a) is valid.

If we introduce (2.32) directly into (2.1) we obtain after using (2.34) and (2.34a) and upon defining

$$A'^\mu(q) \equiv A^{\mu(e)}(x-q) - A^{\mu(e)}(x), \qquad (2.35)$$

$$P'_\mu \equiv i [p_\mu, \ ] - A'^\mu(q), \qquad (2.36)$$

where

$$P'_\mu U \equiv i [p_\mu, U] - A'(q) U \qquad (2.37)$$

$$P'_\mu P'^\mu \langle x-q | A^\nu | x \rangle - P'_\mu P'^\nu \langle x-q | A^\mu | x \rangle = 0. \qquad (2.38)$$

(2.38) is entirely equivalent to (2.21) and the structure of the equations
(2.38) would lead us to make the same remarks (section 5) regarding
(2.38) as were made regarding the structure of (2.21).

III.   THE FREE NON-LOCAL ELECTROMAGNETIC FIELD

7.   Variation Principle for Field Equations.

The field equations for the electromagnetic field may be obtained by considering the trace of the operator[5]

$$L = - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{3.1}$$

where in this section

$$F_{\mu\nu} \equiv i [p_\mu, A_\nu] - i [p_\nu, A_\mu] \tag{3.2}$$

If we proceed in the manner of a previous report[6],[3] we can show that if the field equations

$$i [p_\mu, F^{\mu\nu}] = 0, \tag{3.3}$$

are satisfied, then the vector operator

$$N^\mu = - \tfrac{i}{2} [A_\nu, F^{\mu\nu}], \tag{3.4}$$

satisfies the conservation equations

$$[p_\mu, N^\mu] = 0. \tag{3.5}$$

Furthermore if (3.3) is satisfied then the symmetric stress energy momentum tensor

$$T^{\mu\nu} = T^{\nu\mu} \equiv \tfrac{1}{4}\left(\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\alpha}F_{.\alpha}^{\nu} - F_{.\alpha}^{\nu}F^{\mu\alpha}\right.$$

$$\left. - F^{\alpha\mu}F_{\alpha.}^{\nu} - F_{\alpha.}^{\nu}F^{\alpha\mu}\right), \tag{3.6}$$

satisfies the conservation equations

$$[p_\mu, T^{\mu\nu}] = 0. \tag{3.7}$$

8. Solution of Free Field Equations and Second Quantization in Canonical Terms.

The solution of (3.3) which is hermitian may be expressed in terms of our basic functions as

$$A^\mu = \int |k|^{-1}\left[A^\mu(q,k^+)e^{iqp}e^{ik^+x} + \tilde{A}(q,+k)e^{-ik^+x}e^{-iqp}\right]d^3k\,d^4q \tag{3.8}$$

where

$$f(q,k^{\pm}) \equiv f(q,k_1,k_2,k_3,\pm|k|), \tag{3.9}$$

and

$$k^+_\mu A^\mu (q, k^+) = 0, \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\}$$
$$k^+_\mu \tilde{A} (q, + k^+) = 0, \qquad (3.10)$$

to be consistent with the supplementary condition

$$[p_\mu, A^\mu] = 0. \qquad (3.11)$$

If we anticipate second quantization according to the commutation relationships

$$[\tilde{A}^r(q', k'^\pm), A^s(q'', k''^\pm)] = (8\pi^3)^{-1} |k'| \varepsilon (q') \delta (q'-q'') \times$$
$$\delta_3 (k'-k'') \eta^{rs}, \qquad (3.12)$$

where $r, s = 1, 2, 3$, $\eta^{rs} = 0$, $r \neq s$; $= -1$, $r = s$ and $\varepsilon (q')$ is for the moment an arbitrary function of the c-numbers q^μ, and with all the other A's commuting we will discover that the terms involving the displacement operators p in (3.8) may be transformed away by means of a suitable unitary transformation. Let us consider the operator (hermitian)

$$\phi \equiv \int \propto p_\mu \, q^\mu [\tilde{A}^r(q, k^+) A_r (q, p^+)] \, d^3 k \, d^4 q \qquad (3.13)$$

From (3.12) we can conclude that

$$\left.\begin{aligned}
[\phi, A^3(q, k^+)] &= \alpha |k| \varepsilon(q) p_\mu q^\mu / 8\pi^3 A^s(q, k^+), \\
[\phi, \tilde{A}^s(q, +k^+)] &= -\alpha |k| \varepsilon(q) p_\mu q^\mu / 8\pi^3 \tilde{A}^s(q, +k^+).
\end{aligned}\right\} \tag{3.14}$$

(3.14) implies that with

$$S \equiv e^{i\phi}, \tag{3.15}$$

$$\left.\begin{aligned}
S A^s(q, k^+) S^{-1} &= e^{i\alpha |k| \varepsilon(q) p_\mu q^\mu / 8\pi^3 A^s(q, k^+)} \\
S \tilde{A}^s(q, +k^+) S^{-1} &= e^{-i\alpha |k| \varepsilon(q) p_\mu q^\mu / 8\pi^3 \tilde{A}^s(q, +k^+)}
\end{aligned}\right\} \tag{3.16}$$

Consequently, if α is chosen to have the value α_1 given by

$$\alpha_1 = -8\pi^3 |k|^{-1} \varepsilon^{-1}(q), \tag{3.17}$$

then if we denote by S_1 and ϕ_1 the expressions (3.15) and (3.13) with the α occurring therein replaced by α_1 given by (3.17), (3.16) becomes

$$S_1 \Lambda^s(q, k^+) S_1^{-1} = e^{-i P_\mu q^\mu} A^s(q, k^+) \Bigg\}$$

$$S_1 \tilde{A}^s(q, +k^+) S_1^{-1} = e^{i P_\mu q^\mu} \tilde{A}^s(q, +k^+). \Bigg\} \quad (3.18)$$

However, we must note that as a consequence of our choice for S_1

$$S_1 \chi^\mu \cdot S_1^{-1} =$$

$$\chi^\mu + \int \alpha_1 q^\mu \left[\tilde{A}^r(q, k^+) A_r(q, k^+) \right] d^3k \, d^4q, \quad (3.19)$$

which may be verified upon recognizing that P_μ and χ^μ commute with the rest of the symbols. α_1 is given by (3.17). (3.19) may be written as

$$S_1 \chi^\mu S_1^{-1} = \chi^\mu + Q^\mu, \quad (3.20)$$

where

$$Q^\mu \equiv \int \alpha_1 q^\mu \left[\tilde{A}^r(q, k^+) A_r(q, k^+) \right] d^3k \, d^4q. \quad (3.21)$$

If we now invoke the equations (3.18), (3.19) and (3.20) we discover that

$$S_1 A^s S_1^{-1} = \int |k|^{-1} \Big[A^s(q, k^+) e^{i k_\mu^+ q^\mu} e^{i k^+ x} +$$

$$\widetilde{A}(q, +k^+) e^{-i k_\mu^- q^\mu} e^{-i k_\mu^+ q^\mu} e^{-i k_\mu^- x^-} \Big] d^3 k \, d^4 q. \tag{3.22}$$

From (3.10) and (3.11) we obtain

$$A^\circ = -\int |k|^{-2} k_s \Big[A^s(q, k^+) e^{i q \cdot p} e^{i k^+ x} + \widetilde{A}^s(q, +k^+) e^{-i k^+ x} e^{-i q \cdot p} \Big] d^3 k \, d^4 q \tag{3.23}$$

and correspondingly in view of (3.18) and (3.20)

$$S_1 A^\circ S_1^{-1} = -\int |k|^{-2} k_s \Big[A^s(q, k^+) e^{i k_\mu^+ q^\mu} e^{i k^+ x} +$$

$$\widetilde{A}^s(q, +k^+) e^{-i k_\mu^+ q^\mu} e^{-i k_\mu^+ q^\mu} e^{-i k_\mu^+ x^\mu} \Big] d^3 k \, d^4 q. \tag{3.24}$$

Now if we denote by ϕ_2^+ the expression

$$\phi_2^+ \equiv k_\mu^+ Q^\mu, \tag{3.25}$$

where Q^μ is given by (3.21) we can show in a manner similar to that which led to (3.18) that

$$\left. \begin{array}{l} S_2^+ A^s(q, k'^+)(S_2^+)^{-1} = e^{-i k_\mu^+ q^\mu} A^s(q, k'^+), \\[2mm] S_2^+ \widetilde{A}^s(q, +k'^+)(S_2^+)^{-1} = e^{i k_\mu^- q^\mu} \widetilde{A}^s(q, +k'^+), \end{array} \right\} \tag{3.26}$$

where

$$S_2^+ \equiv e^{ik_\mu^+ q^\mu} \tag{3.27}$$

By virtue of (3.26) and (3.27), (3.22) and (3.24) may be written as

$$S_1 A S_1^{-1} = \int |k|^{-1} \Big[A^s(q, k^+) e^{ik_\mu^+ q^\mu} e^{ik^+ x} + e^{-ik_\mu^+ q^\mu} \widetilde{A}^s(q, +k^+) e^{-ik_\mu^+ x^\mu} \Big] d^3k \, d^4q, \tag{3.28}$$

$$S_1 A^0 S_1^{-1} = -\int |k|^{-2} k_s \Big[\dot{A}^s(q, k^+) e^{ik_\mu^+ q^\mu} e^{ik^+ x} - e^{-ik_\mu^+ q^\mu} \widetilde{A}^s(q, +k^+) e^{-ik_\mu^+ x^\mu} \Big] d^3k \, d^4q. \tag{3.29}$$

(3.28) and (3.29) enable us to state that the transformed vector potentials have been localized in the sense of

$$[x^r, S_1 A^\mu S_1^{-1}] = 0. \tag{3.30}$$

(3.28) and (3.29) may be written as

$$S_i A^s S_i^{-1} = \int |k|^{-1} \left[A^s(q, k) e^{ik_\mu q^\mu} \delta(k_0 - |k|) + \right.$$

$$\left. e^{ik_\mu q^\mu} \tilde{A}^s(q, k) \delta(k_0 + |k|) \right] e^{ik_\mu x^\mu} d^4k \, d^4q, \quad (3.31)$$

and

$$S_i A^0 S_i^{-1} = - \int |k|^{-2} k_s \left[A^s(q, k) e^{ik_\mu q^\mu} \delta(k_0 - |k|) - \right.$$

$$\left. e^{ik_\mu q^\mu} \tilde{A}^s(q, k) \delta(k_0 + |k|) \right] e^{ik_\mu z^\mu} d^4k \, d^4q, \quad (3.32)$$

where $\delta(k_0 \pm |k|)$ is the one dimensional Dirac function of the indicated arguments. In order to preserve hermiticity we must impose the condition that

$$A^s(q, -k) = A^s(q, +k), \quad (3.33)$$

$$\tilde{A}^s(q, -k) = \tilde{A}^s(q, +k),$$

9. **Calculation of Energy-Momentum Vector and Number of Quanta Operator.**

In this section the A^μ and $F_{\mu\nu}$ which appear will be assumed to be the transformed expressions $S_i A^\mu S_i^{-1}$ and $S_i F_{\mu\nu} S_i^{-1}$ where S_i is defined after (3.17). Furthermore

we shall denote by $A^s(k)$ the expression

$$A^s(k) = \int A^s(q, k) \, d^4q, \tag{3.34}$$

so that remembering our convention at the beginning of this section (3.31) and (3.32) may be written as

$$A^s = \int |k|^{-1} \left[A^s(k) e^{ik_\mu q^\mu} \delta(k_0 - |k|) + e^{ik_\mu q^\mu} \widetilde{A}^s(k) \delta(k_0 + |k|) \right] \times$$
$$e^{ik_\mu x^\mu} \, d^4k, \tag{3.35}$$

$$A^0 = -\int |k|^{-2} k_s \left[A^s(k) e^{ik_\mu q^\mu} \delta(k_0 - |k|) + e^{ik_\mu q^\mu} \widetilde{A}^s(k) \delta(k_0 + |k|) \right] \times$$
$$e^{ik_\mu x^\mu} \, d^4k. \tag{3.36}$$

If we now introduce (3.35) and (3.36) into (3.2) we obtain:

$$F_{\mu\nu} = \int |k|^{-1} \left[F_{\mu\nu}(k) e^{ik_\mu q^\mu} \delta(k_0 - |k|) + e^{ik_\mu q^\mu} \widetilde{F}_{\mu\nu}(k) \delta(k_0 + |k|) \right] \times$$
$$e^{ik_\mu x^\mu} \, d^4k, \tag{3.37}$$

where

$$F_{os}(k) \equiv i\left[k_o A_s(k) - k_s A_o(k)\right] = -E_s(k),$$

$$\widetilde{F}_{os}(k) \equiv i\left[k_o \widetilde{A}_s(k) - k_s \widetilde{A}_o(k)\right] = -\widetilde{E}_s(k),$$

(3.38)

$$F_{rs}(k) \equiv i\left[k_r A_s(k) - k_s A_r(k)\right],$$

$$\widetilde{F}_{rs}(k) \equiv i\left[k_r \widetilde{A}_s(k) - k_s \widetilde{A}_r(k)\right].$$

(3.39)

But according to the usual identification $F_{os} = -E_s$, $F_{12} = -H_3$, $F_{23} = -H_1$ and $F_{31} = -H_2$, (3.38) and (3.39) may be written in vector notation as

$$\underline{H}(k) = -i\, \underline{k} \times \underline{A}(k),$$

$$\widetilde{\underline{H}}(k) = -i\, \underline{k} \times \widetilde{\underline{A}}(k).$$

(3.40)

$$\underline{E}(k) = -i\left[k_o \underline{A}(k) - \underline{k}\, A_o(k)\right]$$

$$\widetilde{\underline{E}}(k) = -i\left[k_o \widetilde{\underline{A}}(k) - \underline{k}\, \widetilde{A}_o(k)\right]$$

(3.41)

That the $F_{\mu\nu}$ given by equation (3.37) is hermitian may be surmised by recalling our hermiticity condition (3.33) which

implies $F_{\mu\nu}(-k) = -F_{\mu\nu}(k)$. From (3.11) we must have

$$
\left.\begin{aligned}
A_o(k) &= (k_o)^{-1}\, \underset{\sim}{k} \cdot \underset{\sim}{A}(k), \\[2mm]
\tilde{A}_o(k) &= (k_o)^{-1}\, \underset{\sim}{k} \cdot \underset{\sim}{\tilde{A}}(k).
\end{aligned}\right\} \tag{3.42}
$$

If we introduce three mutually orthogonal unit vectors $\underset{\sim}{e}(k),\ \underset{\sim}{e}_1(k),\ \underset{\sim}{e}_2(k)$ which satisfy:

$$
\left.\begin{aligned}
\underset{\sim}{k} \times \underset{\sim}{e}(k) &= 0, \\[2mm]
\underset{\sim}{e}_1(k) \times \underset{\sim}{e}_2(k) &= \underset{\sim}{e}(k), \\[2mm]
\underset{\sim}{e}_2(k) \times \underset{\sim}{e}(k) &= \underset{\sim}{e}_1(k), \\[2mm]
\underset{\sim}{e}(k) \times \underset{\sim}{e}_1(k) &= \underset{\sim}{e}_2(k),
\end{aligned}\right\} \tag{3.43}
$$

then

$$
\underset{\sim}{A}(k) = \left(\underset{\sim}{e}(k)\cdot\underset{\sim}{A}(k)\right)\underset{\sim}{e}(k) + \left(\underset{\sim}{e}_1(k)\cdot\underset{\sim}{A}(k)\right)\underset{\sim}{e}_1(k)
$$
$$
+ \left(\underset{\sim}{e}_2(k)\cdot\underset{\sim}{A}(k)\right)\underset{\sim}{e}_2(k), \tag{3.44}
$$

and from (3.42)

$$A_o(k) = (k_o)^{-1} |k| \, \underset{\sim}{e}(k) \cdot A(k), \tag{3.45}$$

with similar expressions for $\widetilde{\underset{\sim}{A}}(k)$ and $\widetilde{A}_o(k)$. Consequently,

$$\left. \begin{aligned} \underset{\approx}{E}(k) \cdot \underset{\sim}{e}(k) &= 0, \\ \underset{\approx}{E}(k) \cdot \underset{\sim}{e}_1(k) &= -i k_o \left(\underset{\sim}{e}_1(k) \cdot \underset{\sim}{A}(k) \right), \\ \underset{\approx}{E}(k) \cdot \underset{\sim}{e}_2(k) &= -i k_o \left(\underset{\sim}{e}_2(k) \cdot \underset{\sim}{A}(k) \right), \end{aligned} \right\} \tag{3.46}$$

from (3.41), (3.44), (3.45) and the properties of $\underset{\sim}{e}, \underset{\sim}{e}_1, \underset{\sim}{e}_2$. (3.46) implies that we can write

$$\underset{\approx}{E}(k) = -i k_o \left(\underset{\sim}{e}_1(k) \cdot \underset{\sim}{A}(k) \right) \underset{\sim}{e}_1(k) - i k_o \left(\underset{\sim}{e}_2(k) \cdot \underset{\sim}{A}(k) \right) \underset{\sim}{e}_2(k). \tag{3.47}$$

Also,

$$\underline{H}(k) \cdot \underline{e}(k) = 0,$$

$$\underline{H}(k) \cdot \underline{e}_1(k) = i|k|\underline{A}(k) \cdot \underline{e}_2(k),$$

$$\underline{H}(k) \cdot \underline{e}_2(k) = -i|k|\underline{A}(k) \cdot \underline{e}_1(k),$$

(3.43)

from (3.40), (3.44) and the properties of the unit vectors. Hence

$$\underline{H}(k) = i|k|\left(\underline{e}_2(k) \cdot \underline{A}(k)\right)\underline{e}_1(k) - i|k|\left(\underline{e}_1(k) \cdot \underline{A}(k)\right)\underline{e}_2(k)$$

$$= -|k|(k_0)^{-1}\underline{e}(k) \times \underline{E}(k)$$

(3.49)

The expressions for $\widetilde{\underline{E}}(k)$ and $\widetilde{\underline{H}}(k)$ are quite similar to those given by (3.47) and (3.49).

Using (3.38), (3.39) and (3.49) we may write

$$\underline{E} = \int |k|^{-1}\left[\underline{E}(k)e^{ik_\mu q^\mu}\delta(k_0 - |k|) + e^{ik_\mu q^\mu}\widetilde{\underline{E}}(k)\delta(k_0 + |k|)\right] \times$$

$$e^{ik_\mu x^\mu}d^4k,$$

(3.50)

$$\underline{H} = \int (k_0)^{-1}\left[\underline{e}(k) \times \underline{E}(k)e^{ik_\mu q^\mu}\delta(k_0 - |k|) + e^{ik_\mu q^\mu}\underline{e}(k) \times \widetilde{\underline{E}}(k) \times\right.$$

$$\left.\delta(k_0 + |k|)\right]e^{ik_\mu x^\mu}d^4k,$$

(3.51)

From (3.6) we have

$$T^{oo} = \tfrac{1}{4}\left(F^{\alpha\beta} F_{\alpha\beta} - F^{os} F^{o}_{\cdot s} - F^{o}_{\cdot\alpha} F^{o\alpha} - F^{\alpha o} F^{o}_{\alpha\cdot} - F_{\alpha\cdot} F^{\alpha o} \right),$$

$$= \tfrac{1}{4}\left(2F^{os} F_{os} + F^{rs} F_{rs} - F^{os} F^{o}_{\cdot s} - F^{o}_{\cdot s} F^{os} - F^{so} F^{o}_{s\cdot} - F^{o}_{s\cdot} F^{so} \right),$$

$$= \tfrac{1}{4}\left(-2\,\underset{\sim}{E}\cdot\underset{\sim}{E} + 2\,\underset{\sim}{H}\cdot\underset{\sim}{H} + \underset{\sim}{E}\cdot\underset{\sim}{E} + \underset{\sim}{E}\cdot\underset{\sim}{E} + \underset{\sim}{E}\cdot\underset{\sim}{E} + \underset{\sim}{E}\cdot\underset{\sim}{E} \right),$$

$$T^{oo} = \tfrac{1}{2}\left[\underset{\sim}{E}\cdot\underset{\sim}{E} + \underset{\sim}{H}\cdot\underset{\sim}{H} \right]. \tag{3.52}$$

$$T^{ot} = \tfrac{1}{4}\left(-F^{o\alpha} F^{t}_{\cdot\alpha} - F^{t}_{\cdot\alpha} F^{o\alpha} - F^{\alpha o} F^{t}_{\alpha\cdot} - F^{t}_{\alpha\cdot} F^{\alpha o} \right),$$

$$= \tfrac{1}{4}\left(-F^{os} F^{t}_{\cdot s} - F^{t}_{\cdot s} F^{os} - F^{so} F^{t}_{s\cdot} - F^{t}_{s\cdot} F^{so} \right),$$

$$= \tfrac{1}{4}\left(E_s F_{ts} + F_{ts} E_s + E_s F_{ts} + F_{ts} E_s \right),$$

$$= \tfrac{1}{2}\left(E_s F_{ts} + F_{ts} E_s \right),$$

$$T_{ot} = \tfrac{1}{2}\left[(\underset{\sim}{E}\times\underset{\sim}{H})_t - (\underset{\sim}{H}\times\underset{\sim}{E})_t \right]. \tag{3.53}$$

From (3.4)

$$N^o = N_o = -\frac{1}{2}[A_s, F^{os}] = \frac{i}{2}[A_s, F_{os}],$$

$$= -\frac{i}{2}[\underset{\sim}{A} \cdot \underset{\sim}{E} - \underset{\sim}{E} \cdot \underset{\sim}{A}], \tag{3.54}$$

$$N^s = -N_s = -\frac{i}{2}[A_\nu, F^{s\nu}] = -\frac{i}{2}[A_o, F^{so}] - \frac{i}{2}[A_t, F^{st}]$$

$$= \frac{i}{2}[A_o, F_{so}] - \frac{i}{2}[A_t, F_{st}], \quad \text{or}$$

$$\underset{\sim}{N} = \frac{i}{2}[A_o \underset{\sim}{E} - \underset{\sim}{E} A_o] + \frac{i}{2}[\underset{\sim}{A} X \underset{\sim}{H} - \underset{\sim}{H} X \underset{\sim}{A}]. \tag{3.55}$$

In developing (3.52), (3.53), (3.54) and (3.55) we used the convention that Greek indices run from 0 to 3 and Latin indices from 1 to 3. The indices were raised or lowered using the flat space metric tensor $\eta^{\mu\nu}$ or $\eta_{\mu\nu}$ and the customary relations between the F's and the E, H's were used together with some elementary vector analysis.

Since the χ^μ commute with all the rest of the operators, we need not appeal to the device developed in some earlier work[2] for integrating non-local functions. We may regard the χ^μ as C-numbers. If we introduce (3.50) and (3.51) into (3.52) and form $\int T^{oo} d^3\chi$ we obtain after making use of certain vector identities

$$G^0 \equiv \int T^{00} \, d^3x$$

$$= 4\pi^3 \int \dot{\delta}_3(k+k') \left[|k|^{-2} - (k_o \, k'_o)^{-1} \right] \times$$

$$\left[\underset{\sim}{E}(k) e^{ik_\mu q^\mu} \cdot \underset{\sim}{E}(k') e^{ik'_\mu q^\mu} \delta(k_o - |k|) \delta(k'_o - |k|) + \right.$$

$$e^{ik_\mu q^\mu} \underset{\sim}{\breve{E}}(k) \cdot \underset{\sim}{E}(k) e^{ik'_\mu q^\mu} \delta(k_o + |k|) \delta(k'_o + |k|) +$$

$$\underset{\sim}{E}(k) e^{ik_\mu q^\mu} \cdot e^{ik'_\mu q^\mu} \underset{\sim}{E}(k') \delta(k_o - |k|) \delta(k'_o + |k|) +$$

$$\left. e^{ik_\mu q^\mu} \underset{\sim}{\widetilde{E}}(k) \cdot \underset{\sim}{E}(k') e^{ik'_\mu q^\mu} \delta(k_o + |k|) \delta(k'_o - |k|) \right] \times$$

$$e^{i(k_o + k'_o) x^o} \, d^4k \, d^4k', \tag{3.56}$$

after recalling that

$$\int e^{i(k_s + k'_s) x^s} \, d^3x = 8\pi^3 \, \delta_3(k+k'), \tag{3.57}$$

where $\delta_3(k+k')$ is the three dimensional Dirac function of the indicated arguments. (3.56) may be simplified after using the properties of the Dirac functions appearing therein to

$$G^\circ = 8\pi^3 \int |k|^{-2} \delta(k+k') \left[\underset{\sim}{E}(k) \widetilde{\underset{\sim}{E}}(k') \delta(k_0 - |k|) \right.$$

$$\left. + e^{ik_\mu \varphi^\mu} \widetilde{\underset{\sim}{E}}(k) \underset{\sim}{E}(k') e^{-ik_\mu \varphi^\mu} \delta(k_0 + |k|) \right] \times$$

$$d^4k \, d^4k'. \tag{3.58}$$

$$= -8\pi^3 \int |k|^{-2} \left[\underset{\sim}{E}(k^+) \cdot \widetilde{\underset{\sim}{E}}(k^+) + e^{ik_\mu \varphi^\mu} \widetilde{\underset{\sim}{E}}(k^-) \cdot \underset{\sim}{E}(k^-) e^{-ik_\mu \varphi^\mu} \right] \times$$

$$d^3k \tag{3.59}$$

since $\underset{\sim}{E}(k) = -\underset{\sim}{E}(-k)$; $\widetilde{\underset{\sim}{E}}(k) = -\widetilde{\underset{\sim}{E}}(-k)$.

If we introduce (3.47) into (3.59) we obtain

$$G^\circ = 8\pi^3 \sum_{i=1}^{3} \int \left[\left(\underset{\sim}{e}_i(k) \cdot \underset{\sim}{A}(k^+) \right) \left(\underset{\sim}{e}_i(k) \cdot \widetilde{\underset{\sim}{A}}(k^+) \right) \right.$$

$$\left. + e^{ik_\mu \varphi^\mu} \left(\underset{\sim}{e}_i(k) \cdot \widetilde{\underset{\sim}{A}}(k^-) \right) \left(\underset{\sim}{e}_i(k) \cdot \underset{\sim}{A}(k^-) \right) e^{-ik_\mu \varphi^\mu} \right] \times$$

$$d^3k. \tag{3.60}$$

In the same fashion we can calculate the momentum vector $\underset{\sim}{G}$ by evaluating from (3.53)

$$\underset{\sim}{G} = \int T_{ot} \, d^3x$$

$$= \frac{1}{2} \int \left[(\underset{\sim}{E} \times \underset{\sim}{H}) - (\underset{\sim}{H} \times \underset{\sim}{E}) \right] d^3x$$

$$= 8\pi^3 \int |k|^{-3} \underset{\sim}{k} \Big[\underset{\sim}{E}(k) \cdot \underset{\sim}{\widetilde{E}}(k) \, \delta(k_0 - |k|) -$$

$$e^{ik_\mu \varphi^\mu} \underset{\sim}{\widetilde{E}}(k) \cdot \underset{\sim}{E}(k) e^{-ik_\mu \varphi^\mu} \delta(k_0 + |k|) \Big] d^4k . \qquad (3.61)$$

(3.59) and (3.61) may be combined into the following expression for the 4 - vector G^μ:

$$G^\mu =$$

$$-8\pi^2 \int k^\mu |k|^{-3} \Big[\underset{\sim}{E}(k) \cdot \underset{\sim}{\widetilde{E}}(k) \delta(k_0 - |k|) -$$

$$e^{ik_\mu \varphi^\mu} \underset{\sim}{\widetilde{E}}(k) \cdot \underset{\sim}{E}(k) e^{-ik_\mu \varphi^\mu}_{\delta(k_0 + |k|)} \Big] d^4k \qquad (3.62)$$

$\underset{\sim}{E}(k)$ and $\underset{\sim}{\widetilde{E}}(k)$ are given by (3.59).

Let us now proceed to calculate the number of Quanta Operator N which is defined to be

$$N \equiv \int N^0 \, d^3x, \qquad (3.62')$$

where N° is given by (3.54). Now

$$N_o = -\tfrac{i}{2} [\underset{\sim}{A} \cdot \underset{\sim}{E} - \underset{\sim}{E} \cdot \underset{\sim}{A}],$$

so that from (3.44), (3.47), (3.50) and (3.35),

$$N_o = -\tfrac{1}{2} \int |k|^{-1} |k'|^{-1} \left\{ \left[\underset{\sim}{A}(k) e^{ik_\mu q^\mu} \delta(k_o - |k|) + e^{ik_\mu q^\mu} \underset{\sim}{\widetilde{A}}(k) \delta(k_o + |k|) \right] - \right.$$
$$\left[\underset{\sim}{E}(k) e^{ik_\mu q^\mu} \delta(k'_o - |k'|) + e^{ik'_\mu q^\mu} \underset{\sim}{\widetilde{E}}(k) \delta(k'_o + |k'|) \right] -$$
$$\left[\underset{\sim}{E}(k) e^{ik_\mu q^\mu} \delta(k_o - |k|) + e^{ik_\mu q^\mu} \underset{\sim}{\widetilde{E}}(k) \delta(k_o + |k|) \right].$$
$$\left. \left[\underset{\sim}{A}(k) e^{ik'_\mu q^\mu} \delta(k'_o - |k|) + e^{ik'_\mu q^\mu} \underset{\sim}{\widetilde{A}}(k') \delta(k'_o + |k|) \right] \right\} e^{i(k' + k_\mu)x^\mu} d^4k \, d^4k'. \quad (3.63)$$

If we insert (3.63) into (3.62) we obtain

$$N = 4\pi^3 i \int |k|^{-2} \left\{ \left[-\underset{\sim}{A}(k) \cdot \underset{\sim}{\widetilde{E}}(k) \delta(k_o - |k|) - e^{ik_\mu q^\mu} \underset{\sim}{\widetilde{A}}(k) \cdot \underset{\sim}{E}(k) e^{-ik_\mu q^\mu} \delta(k_o + |k|) \right] \right.$$
$$\left. + \left[\underset{\sim}{E}(k) \cdot \underset{\sim}{\widetilde{A}}(k) \delta(k_o - |k|) + e^{ik_\mu q^\mu} \underset{\sim}{\widetilde{E}}(k) \cdot \underset{\sim}{A}(k) e^{-ik_\mu q^\mu} \delta(k_o + |k|) \right] \right\} \times$$
$$d^4k. \quad (3.64)$$

But from (3.47) and (3.44)

$$\underline{A}(k) \cdot \underline{\tilde{E}}(k) = -i k_o \left(\underline{e}_1(k) \cdot \underline{A}(k) \right)\left(\underline{e}_1 \cdot \underline{\tilde{A}}(k) \right) - i k_o \left(\underline{e}_2(k) \cdot \underline{A}(k) \right)\left(\underline{e}_2(k) \cdot \underline{\tilde{A}}(k) \right)$$

$$= \underline{E}(k) \cdot \underline{\tilde{A}}(k)$$

$$= i k_o^{-1} \underline{E}(k) \cdot \underline{\tilde{E}}(k). \tag{3.65}$$

Similarly,

$$\underline{\tilde{A}}(k) \cdot \underline{E}(k) = \underline{\tilde{E}}(k) \cdot \underline{A}(k) - i k_o^{-1} \underline{\tilde{E}}(k) \cdot \underline{E}(k). \tag{3.66}$$

Consequently, (3.64) becomes

$$N = -8\pi^3 \int |k|^{-3} \Big[\underline{E}(k) \cdot \underline{\tilde{E}}(k)\, \delta(k_o - |k|) - e^{i k_\mu q^\mu} \underline{\tilde{E}}(k) \cdot \underline{E}(k) \times$$

$$e^{-i k_\mu q^\mu}_{\delta(k_o + |k|)} \Big]\, d^4 k, \tag{3.67}$$

for the number of Quanta operator.

The structure of (3.67) and (3.62) suggests that we may liken the operator

$$\rho(k) = -8\pi^3 |k|^{-3} \Big[\underline{E}(k) \cdot \underline{\tilde{E}}(k)\, \delta(k_o - |k|) - e^{i k_\mu q^\mu} \underline{\tilde{E}}(k) \cdot \underline{E}(k)\, e^{-i k_\mu q^\mu}_{\delta(k_o + |k|)} \Big] \tag{3.68}$$

to a density operator in such manner as to indicate that $\rho(k) d^4 k$

gives the net number of Quanta operator in the 4- volume d^4k in k- space. As a consequence of this definition and interpretation (3.67) and (3.62) may be written as

$$G^\mu = \int k^\mu \rho(k)\, d^4k,\qquad (3.69)$$

$$N = \int \rho(k)\, d^4k.\qquad (3.70)$$

From (3.70) we may construct other operators associated with certain c- number functions $f(k)$ say

$$F \equiv \int f(k)\, \rho(k)\, d^4k,\qquad (3.71)$$

which may be interpreted to correspond to the operator which indicates the net contribution of the "particles"' $f(k)$. For example, if $f(k)\sqrt{k_o^2 - \underline{k}\cdot\underline{k}}$ in our case then $F = 0$ because of the appearance of $\delta(k_o \pm |k|)$ as factors in (3.68). However, if $f(k) = k_o |k|^{-1}$ then F would be <u>positive definite</u> and would be interpreted to denote the number of Quanta as contrasted to N in (3.70) which we have interpreted to denote the <u>net</u> number of Quanta. In the former example $f(k)$ clearly corresponded to the mass of the particle associated with the 4- vector k^μ as in the usual interpretation. If we denote by N^+ the operator for the

number of Quanta then from the above discussion

$$N^+ \equiv \int k_0 \, |k_0|^{-1} \, f(k) \, d^4 k . \tag{3.72}$$

The operator for the total mass of the particles comprising the field could be defined as

$$M' = \int \sqrt{k_0^2 - \underset{\sim}{k} \cdot \underset{\sim}{k}} \; f(k) \, d^4 k, \tag{3.73}$$

or

$$M = \int |k_0|^{-1} \, k_0 \sqrt{k_0^2 - \underset{\sim}{k} \cdot \underset{\sim}{k}} \; f(k) \, d^4 k . \tag{3.74}$$

(3.73), if taken as a definition, would in general permit the appearance of negative masses. On the other hand the definition (3.74) would in general be positive because of the positive definite character of the integrand. In any event for the case that we are considering both M and M' make no contribution.

Before we close this section let us record for future use some of the non-vanishing commutators involving $A(q, k^\pm)$, $\widetilde{A}(q, k^\pm)$, $E(k^\pm)$, $\widetilde{E}(k^\pm)$, N and Q^μ. From (3.12) and (3.34) we have

$$\left[\widetilde{A}_r(q,k'), A_s(k')\right] = (8\pi^3)^{-1}|k'|\,\epsilon(q)\,\delta_3(k'-k'')\,\eta_{rs} \qquad (3.75)$$

Upon using (3.40), (3.41) and (3.75) it can be shown that

$$\left[\widetilde{E}_r(k'^\pm), A_s(q,k''^\pm)\right] = \left[\widetilde{A}_r(q,k'^\pm), E_s(k''^\pm)\right]$$

$$= \mp i\left[|k'|^2\eta_{rs} - k'_r k'_s\right]\epsilon(q)\,\delta_3(k'-k''). \qquad (3.76)$$

If we integrate both sides of (3.76) with respect to q we obtain from the definition of $A(k)$ and $\widetilde{A}(k)$: (3.34)

$$\left[\widetilde{E}_r(k'^\pm), A_s(k''^\pm)\right] = \left[\widetilde{A}_r(k'^\pm), E_s(k''^\pm)\right]$$

$$= \mp i\left[|k'|^2\eta_{rs} - k'_r k'_s\right]\delta_3(k'-k'')\int\epsilon(q)\,d^4q\,. \qquad (3.77)$$

In a similar way

$$\left[\widetilde{E}_r(k'^\pm), E_s(k''^\pm)\right] = -|k'|\left[|k'|^2\eta_{rs} - k'_r k'_s\right]\delta_3(k'-k'')\int\epsilon(q)d^4q, \qquad (3.78)$$

$$\left[A_r(q,k^+), N\right] = -8\pi^3\,i\,|k|^{-1}\,\epsilon(q)\,E_r(k^+), \qquad (3.79)$$

$$\left[\widetilde{A}_r(q,k^+), N\right] = -8\pi^3\,i\,|k|^{-1}\,\epsilon(q)\,\widetilde{E}_r(k^+), \qquad (3.80)$$

$$\left[A_r(q,k^-),N\right] = 8\pi^3 i |k|^{-1} \varepsilon(q) e^{ik_\mu q^\mu} E_r(k^-) e^{-ik_\mu q^\mu}, \quad (3.81)$$

$$\left[\widehat{A}_r(q,k^-),N\right] = 8\pi^3 i |k|^{-1} \varepsilon(q) e^{ik_\mu q^\mu} \widetilde{E}_r(k^-) e^{-ik_\mu q^\mu}, \quad (3.82)$$

$$\left[E_r(k^+),N\right] = 8\pi^3 E_r(k^+) \int \varepsilon(q)\, d^4 q, \quad (3.83)$$

$$\left[\widetilde{E}_r(k^+),N\right] = 8\pi^3 \widetilde{E}_r(k^+) \int \varepsilon(q)\, d^4 q, \quad (3.84)$$

$$\left[E_r(k^-),N\right] = -8\pi^3 e^{ik_\mu q^\mu} E_r(k^-) e^{-ik_\mu q^\mu} \int \varepsilon(q) d^4 q, \quad (3.85)$$

$$\left[\widetilde{E}_r,(k^-),N\right] = -8\pi^3 e^{ik_\mu q^\mu} E_r(k^-) e^{-ik_\mu q^\mu} \int \varepsilon(q) d^4 q, \quad (3.86)$$

and finally

$$[N,Q^\mu] = -8\pi^3 \int q^\mu |k| \left[\widetilde{E}_r(k) A_r(q,k) - \widetilde{A}_r(q,k) E_r(k)\right] d^4 q\, d^3 k. \quad (3.87)$$

10. Introduction of Boson Operators.

In this section we will continue to assume that all operators appearing, unless explicitly stated otherwise, are the transformed expressions $S_1 G^\mu S_1^{-1}$, $S_1 N S_1^{-1}$, $\cdots$ which we shall simply denote by G^μ, N, $\cdots$ respectively. S_1 is defined after (3.17). If we denote by $\underset{\sim}{e}_3(k)$ the $\underset{\sim}{e}(k)$ which appears in (3.43) then upon defining

$$A_i'(k) \equiv \underset{\sim}{e}_i(k) \cdot A(k), \qquad i - 1, 2, 3, \tag{3.88}$$

we find from (3.75) after integrating with respect to $\underset{\sim}{q}$ and using the definition (3.34) that

$$\left[\widetilde{A}_r(k^{\pm\prime}), A_s(k^{\pm\prime\prime})\right] = (8\pi^3)^{-1}|k'|\delta_3(k'-k'')\eta_{rs} \times$$

$$\int \varepsilon(q)\,d^4q, \tag{3.89}$$

which implies in view of (3.88) that

$$\left[\widetilde{A}_i^{\pm\prime}(k), A_j'(k^{\pm\prime\prime})\right] = (8\pi^3)^{-1}|k'|\delta_3(k'-k'')\eta_{ij}\int\varepsilon(q)\,d^4q, \tag{3.90}$$

in view of the properties of our unit vectors $\underset{\sim}{e}_i(k)$. Thus the $A_i'(k)$ satisfy the same commutation relationships as the $A_r(k)$. Introducing (3.88) in (3.47) we obtain

$$\underset{\sim}{E}(k^{\pm}) = -i|k|A_1'(k^{\pm})\underset{\sim}{e}_1(k^{\pm}) - i|k|A_2'(k^{\pm})\underset{\sim}{e}_2(k^{\pm}). \tag{3.91}$$

Consequently, (3.67), (3.60) and (3.61) become

$$N = 8\pi^3\sum_{i=1}^{3}\int|k|^{-1}\left\{A_i'(k^+)\widetilde{A}_i'(k^+) - e^{ik_\mu q^\mu}\widetilde{A}_i'(k^-)A_i'(k^-)e^{-ik_\mu q^\mu}\right\}d^3k, \tag{3.92}$$

$$G_{o}=8\pi^3\sum_{i=1}^{2}\int\left\{A'_i(k^+)\widetilde{A}'_i(k^+)+e^{ik_\mu q^\mu}\widetilde{A}'_i(k^-)A_i(k^-)e^{-ik_\mu q^\mu}\right\}d^3k, \quad (3.93)$$

$$G=8\pi^3\sum_{i=1}^{2}\int|k|^{-1}k\left\{A'_i(k^+)\widetilde{A}'_i(k^+)-e^{ik_\mu q^\mu}\widetilde{A}'_i(k^-)A_i(k^-)e^{-ik_\mu q^\mu}\right\}d^3k, \quad (3.94)$$

If we now define

$$A'_i(k^+)\equiv|k|^{1/2}\eta_i(k^+)(8\pi^3)^{-\frac{1}{2}} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\}$$

$$e^{ik_\mu q^\mu}A'_i(k^-)e^{-ik_\mu q^\mu}=|k|^{1/2}\eta_i(k^-)(8\pi^3)^{-\frac{1}{2}} \qquad (3.95)$$

then in order that

$$\left[\widetilde{\eta}_i(k'^{\pm}),\eta_j(k''^{\pm})\right]=\delta_3(k'-k'')\eta_{ij}, \qquad (3.96)$$

we find from (3.89) that the function $\varepsilon\left(q\right)$ which we have introduced in the beginning section (8) must be normalized to unity:

$$\int\varepsilon(q)\,d^4q=1. \qquad (3.97)$$

Whether a suitable invariant function satisfying (3.97) exists or not cannot at the present stage of development be answered. (3.92), (3.93) and (3.94) become in terms of the Boson operators η

$$N = \sum_{i=1}^{4} \int \left[\eta_i(k^+) \widetilde{\eta}_i(k^+) - \widetilde{\eta}_i(k^-) \eta_i(k^-) \right] d^3k,$$

$$G_0 = \sum_{i=1}^{4} \int |k| \left[\eta_i(k^+) \widetilde{\eta}_i(k^+) + \widetilde{\eta}_i(k^-) \eta_i(k^-) \right] d^3k, \qquad (3.98)$$

$$\underset{\sim}{G} = \sum_{i=1}^{4} \int \underset{\sim}{k} \left[\eta_i(k^+) \widetilde{\eta}_i(k^+) - \widetilde{\eta}_i(k^-) \eta_i(k^-) \right] d^3k,$$

(3.98) may be replaced by a sum by introducing the formalism of Dirac[2] where for a function $f(k)$

$$\int f(k) \, d^3k = \sum_k f(k) \, s^{-1}(k), \qquad (3.99)$$

where $s(k)$ is the number of points per unit volume in the neighborhood of any point $\underset{\sim}{k}$. If the continuous Boson operators η and $\widetilde{\eta}$ appearing in (3.98) are replaced by the discrete Boson operators η_{oi} and $\widetilde{\eta}_{oi}$ according to

$$\eta = s^{\pm\frac{1}{2}} \eta_o ,$$
$$\widetilde{\eta} = s^{1/2} \widetilde{\eta}_o , \qquad (3.100)$$

and use is made of (3.99) we find that

$$N = \sum_{i=1}^{2} \sum_{k} \left[\eta_{oi}(k^+) \tilde{\eta}_{oi}(k^+) - \tilde{\eta}_{oi}(k^-) \eta_{oi}(k^-) \right],$$

$$G_0 = \sum_{i=1}^{2} \sum_{k} |k| \left[\eta_{oi}(k^+) \tilde{\eta}_{oi}(k^+) + \tilde{\eta}_{oi}(k^-) \eta_{oi}(k^-) \right], \Biggr\} \quad (3.101)$$

$$\underset{\sim}{G} = \sum_{i=1}^{2} \sum_{k} \underset{\sim}{k} \left[\eta_{oi}(k^+) \tilde{\eta}_{oi}(k^+) - \tilde{\eta}_{oi}(k^-) \eta_{oi}(k^-) \right].$$

However, [2]

$$S_k \delta_{kk'} = \delta_3(k - k'), \quad (3.102)$$

so that in view of (3.96)

$$\left[\tilde{\eta}_{oi}(k'^{\pm}), \eta_{oj}(k''^{\pm}) \right] = \delta_{k'k''} \eta_{ij}. \quad (3.103)$$

Thus from the well known properties of the Boson operators we conclude that the characteristic values of N are comprised of all positive or negative integers. G_o is positive definite while $\underset{\sim}{G}$ is not. The characteristic values of G_o and $\underset{\sim}{G}$ are sums of integer multiples of $|k|$ and $\underset{\sim}{k}$ respectively. Since the symbols used in this section are the transformed symbols involving the unitary operator S, defined after (3.17) and since the characteristic values are unaltered by unitary transformations the untransformed symbols G^μ and N have the same characteristic values.

IV. INTERACTIONS

11. Introduction.

In this section we will attempt to recast the generalized Schroedinger equation for unitary transformations to a form such as to yield directly and independently of the commutation relationships satisfied or postulated for the fields the result that if the "field equations" are satisfied then S = const. In the usual formulation of the theory a Lagrangian function L appears in the generalized Schroedinger equation as

$$i \frac{\delta S(\sigma)}{\delta \sigma} = L S(\sigma),$$
(4.0)

where $S(\sigma)$ is a unitary operator and $\frac{\delta}{\delta \sigma}$ denotes the functional derivative of a functional of the space-time surface. [7] If L in (4.0) is a "free" field Lagrangian, it is not at all apparent that S = const. unless one undertakes to make further assumptions regarding the commutation relationships and/or appeals to representations of one kind or another as in the usual arguments. [8] It is believed that the Lagrangian function which appears in (4.0) should be constructed in a manner which renders manifest the statement S = const. for free fields.

It would be possible within the framework of the classical theory of $C-$ number fields to arrange for a modified L to vanish. For example, the Lagrangian

$$L = \eta^{\mu\nu} \frac{\partial U}{\partial x^\mu} \frac{\partial U}{\partial x^\nu} ,$$

could be written as

$$L = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} U \frac{\partial U}{\partial x^\nu} - \eta^{\mu\nu} U \frac{\partial^2 U}{\partial x^\mu \partial x^\nu} ,$$

which involves a term of the form of a divergence which presumably makes no contribution. The original Lagrangian L and the modified Lagrangian L' :

$$L' = -\eta^{\mu\nu} U \frac{\partial^2 U}{\partial x^\mu \partial x^\nu} ,$$

would yield the same field equation upon insertion into the classical action principle. Moreover, it would be possible to construct a stress-energy momentum tensor corresponding to L' via the methods of general relativity[9] /or the usual canonical formalism_7 upon replacing L' by its covariant generalization

$$L' = -g^{\mu\nu} U U_{;\mu\nu} ,$$

where $g^{\mu\nu}$ is the metric tensor and $(\ )_{;\mu\nu}$ denotes covariant differentiation relative to the metric $g_{\mu\nu}$. This however is of no import to us at the present time. It is clear that the insertion of a symmetrized L' say

$$L' = -\tfrac{1}{2}\eta^{\mu\nu}\,U\frac{\partial^2 U}{\partial x^\mu \partial x^\nu} = -\tfrac{1}{2}\eta^{\mu\nu}\frac{\partial^2 U}{\partial x^\mu \partial x^\nu}\,U$$

into (4.0) would yield the desired property $S' = $ const. if the field equations are satisfied (in our case $\Box^2 U = 0$). It would be possible in this manner to obtain modified "free" field Lagrangians for all of the fields of current interest. The above procedure, however, suffers a defect in that it is not independent of the manner in which the classical fields, which are promoted to be operators in Quantum Theory, are represented. In order to make some progress in this direction in the next part of our investigation, we shall assume from the outset that the symbols entering our equations are operators and for simplicity we will take our fields to be local.

12. Transformation Theory.

Let us consider the operator Lagrangian L which is a function of some fields f_A, $A = 1, 2, \cdots$, the displacement operators p_μ, $\mu = 0, 1, 2, 4$ and some other given functions of the space-time operators x_μ, say g_β. The consideration of the first

variation of the trace of L expressed in terms of the variation of f_A, p_μ and g_B leads to the expression

$$\delta \operatorname{Tr} L =$$

$$\operatorname{Tr} \left(F^A \delta f_A + N^\mu \delta p_\mu + M^B \delta g_B \right), \tag{4.1}$$

which, if the variations arise from an arbitrary infinitesmal similarity transformation, implies the identity

$$[f_A, F^A] + [p_\mu, N^\mu] + [g_B, M^B] \equiv 0. \tag{4.2}$$

Because of (4.2) we may write

$$\mathcal{L}' \equiv f_A F^A + p_\mu N^\mu + g_B M^B =$$

$$F^A f_A + N^\mu p_\mu + M^B g_B, \tag{4.3}$$

which we will define to be our modified Lagrangian. If (4.3) is constructed so that $\mathcal{L}$ is hermitian, then if we consider our fields to be local the functional equation

$$i \frac{\delta S(\sigma)}{\delta \sigma} \equiv \mathcal{L}' S(\sigma), \tag{4.4}$$

with $\mathcal{L}'$ given by (4.3) defines a unitary operator S'. Now since we are assuming that g_B is a given local function then the expression $[g_B, M|^B] = 0$ in (4.2) so that we need not be concerned with the expressions involving j_B in (4.3). With this taken into consideration we will define

$$\mathcal{L} \equiv f_A F^A + p_\mu N^\mu = F^A f_A + N^\mu p_\mu \qquad (4.5)$$

and insert it in place of $\mathcal{L}'$ which appears in (4.4).

We will now show that if the field equations are satisfied

$$F^A = 0, \qquad (4.6)$$

then $S(\sigma) = $ const. In order to show this let us first rewrite (4.4) with $\mathcal{L}'$ replaced by $\mathcal{L}$ as

$$P(\sigma) S(\sigma) = \mathcal{L} S(\sigma), \qquad (4.7)$$

where in a representation with $\varkappa$ diagonal

$$\langle \varkappa' | P(\sigma) | \varkappa'' \rangle \equiv i \frac{\delta}{\delta\sigma(\varkappa')} \, \delta(\varkappa' - \varkappa') = i \langle \varkappa' | \frac{\delta}{\delta\sigma(\varkappa)} | \varkappa'' \rangle \qquad (4.8)$$

where $\delta(\varkappa' - \varkappa'')$ is the four dimensional Dirac function and it is to

be understood that $\langle \varkappa' | P(\sigma) | \varkappa'' \rangle$ is defined in connection with other functionals of the surface σ . Now

$$\langle \varkappa' | P(\sigma) A(\sigma) | \varkappa'' \rangle = i \int \frac{\delta}{\delta\sigma(\varkappa)} \, \delta(\varkappa' - \varkappa''') \langle \varkappa''' | A(\sigma) | \varkappa'' \rangle d^4 \varkappa'''$$

$$= i \frac{\delta}{\delta\sigma(\varkappa)} \langle \varkappa' | A(\sigma) | \varkappa'' \rangle. \tag{4.9}$$

Also

$$\langle \varkappa' | A(\sigma) P(\sigma) | \varkappa'' \rangle = i \int \langle \varkappa' | A(\sigma) | \varkappa''' \rangle \frac{\delta}{\delta\sigma(\varkappa''')} \delta(\varkappa''' - \varkappa'') d^4 \varkappa''' \tag{4.10}$$

upon introducing (4.8) and using the rules of matrix multiplication.

But

$$\int_{\sigma_o}^{\sigma_1} A(\sigma) \frac{\delta}{\delta\sigma(\varkappa)} B \, d^4 \varkappa = \left[A(\sigma) B \right]_{\sigma_1} - \left[A(\sigma) B \right]_{\sigma_o}$$

$$- \int_{\sigma_o}^{\sigma_1} B \frac{\delta A}{\delta\sigma(\varkappa)} d^4 \varkappa$$

where $\left[\ \right]_\sigma$ denotes the contribution of the argument upon evaluation on σ . Applying this result to (4.10) we obtain

$$\langle \varkappa' | A(\sigma) P(\sigma) | \varkappa'' \rangle = -i \frac{\delta}{\delta\sigma(\varkappa'')} \langle \varkappa' | A(\sigma) | \varkappa'' \rangle, \tag{4.11}$$

owing to the circumstance that the evaluations of $[\ ]_\sigma$ for this case vanish.   (4.9) and (4.11) are such as to indicate that $P(\sigma)$ is an hermitian operator.   For upon introducing $A(\sigma) = 1$ we discover that

$$\langle x' | P(\sigma) | x'' \rangle = \langle x'' | P(\sigma) | x' \rangle^*, \qquad (4.12)$$

upon recalling that

$$\langle x' | x'' \rangle = \langle x'' | x' \rangle = \delta(x' - x'') . \qquad (4.13)$$

If we now take the matrix elements of (4.7) in a representation with x - diagonal we obtain from (4.8) and recalling

$$\langle x' | p_\mu | x'' \rangle = -i \frac{\partial}{\partial x'^\mu} \delta(x' - x''), \qquad (4.14)$$

$$i \frac{\delta}{\delta\sigma(x')} \langle x' | S(\sigma) | x'' \rangle =$$

$$\langle x' | f_A F^A S(\sigma(x)) | x'' \rangle$$

$$-i \int \frac{\partial}{\partial x'^\mu} \delta(x' - x''') \langle x''' | N^\mu(x) S(\sigma(x)) | x'' \rangle d^4 x''' =$$

$$\langle x' | f_A F^A S(\sigma(x)) | x'' \rangle - i \frac{\delta}{\delta\sigma(x')} \int_\sigma \langle x' | N^\mu(x) S(\sigma(x)) | x'' \rangle d\sigma_\mu(x'),$$

since $\frac{\partial}{\partial x^\mu} A(x) = \frac{\delta}{\delta \sigma(x)} \int_{\sigma} A(x') d \sigma_\mu (x')$. Now if N^μ and $S(\sigma(x))$ are presumed diagonal in a representation with $x -$ diagonal

$$\int_{\sigma(x)} \langle x' | N^\mu (x) S(\sigma(x)) | x'' \rangle d \sigma_\mu (x') =$$

$$\langle x' | \int_\sigma N^\mu (x) S(\sigma(x)) d \sigma_\mu (x) | x'' \rangle =$$

$$\langle x' | N(\sigma) S(\sigma(x)) | x'' \rangle , {}^{(*)}$$

where $N(\sigma) \equiv \int_\sigma N^\mu (x) d \sigma_\mu$. $\qquad$ (4. 15)

Hence,

$$-i \frac{\delta}{\delta \sigma(x)} \int_\sigma \langle x' | N^\mu S(\sigma) | x'' \rangle d \sigma_\mu (x') =$$

$$-i \frac{\delta}{\delta \sigma(x')} \langle x' | N(\sigma) S(\sigma) | x'' \rangle ,$$

Therefore,

(*) This replacement is possible only in case $S(\sigma)$ = const. However, we shall define symbolically $\int_\sigma N^\mu S d \sigma_\mu \equiv N S$.

$$i \frac{\delta}{\delta\sigma(z')} \langle x' | S(\sigma) | x'' \rangle =$$

$$\langle x' | f_A F^A S(\sigma) | x'' \rangle - i \frac{\delta}{\delta\sigma(z')} \langle x' | N(\sigma) S(\sigma) | x' \rangle,$$

so that if $F^A = 0$,

$$i \frac{\delta}{\delta\sigma(z)} \left\{ \langle x' | (1 + N(\sigma)) S(\sigma) | x' \rangle \right\} = 0. \qquad (4.16)$$

Remembering that if $F^A = 0$, $N(\sigma)$ is independent of σ
(4.16) possesses a solution of the form $S = S_0'(N)$ of which
a simple unitary form can be exhibited in the form

$$S_0 = e^{i \alpha N}, \qquad (4.17)$$

where α is a real $C -$ number.

In general we would have in the case where both f_A and
F^A are diagonal in the $x-$ representation

$$i \frac{\delta}{\delta\sigma(x)} \left\{ \langle x' | (1 + N(\sigma)) S(\sigma) | x' \rangle \right\} = f_A(x') F^A(x') \langle x' | S(\sigma) | x' \rangle (4.18)$$

whether the field equations are satisfied or not.

(4.18) can also be written as

$$\langle \varkappa' | i \frac{\partial}{\delta \sigma(\varkappa)} [1 + N(\tau)] S(\sigma) | \varkappa'' \rangle =$$

$$\langle \varkappa' | f_A F^A S(\sigma) | \varkappa'' \rangle, \tag{4.19}$$

which would imply that

$$i \frac{\partial}{\delta \sigma(\varkappa)} [1 + N(\sigma)] S(\sigma) = f_A F^A S(\sigma). \tag{4.20}$$

An integral equation corresponding to (4.20) which incorporates the "initial" condition $S(\sigma) = 1$ on σ_c is*

$$S(\sigma) = [1 + N(\sigma)]^{-1} [1 + N(\sigma_0)] - i[1 + N(\sigma)]^{-1} \int_{\sigma_0}^{\sigma} f_A F^A S(\sigma) d^4 \varkappa. \tag{4.21}$$

It is apparent that (4.20) and (4.21) differ from the usual functional equation for the operator $S(\sigma')$ since it involves the expression $N(\sigma)$ defined by (4.15) which from some earlier

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*It is to be noted that $[1 + N(\sigma)]^{-1}$ is symbolic and is defined by
$$U(\sigma) = [1 + N(\sigma)]^{-1} F(\sigma) \quad \text{where } U(\sigma) \text{ satisfies}$$
$$U(\sigma) + \int_{\sigma} N^{\mu}(\varkappa') U[\sigma(\varkappa')] d\sigma_{\mu}(\varkappa') = F(\sigma).$$

work[2] we have defined to be the operator for the net number of Quanta. If we proceed to solve (4.21) it seems to indicate that the matrix elements of S' in the first approximation would involve the reciprocal of the net number of Quanta plus unity so that as the net number of Quanta involved increases the probability for transitions from one state to another would decrease in a manner involving the square of $N(\sigma) + 1$. Qualitatively this finding implies $S(\sigma) \to 1$ as the net number of Quanta increases. Consequently, if our observations are valid, the probability of transitions from one state to another for assemblages which involve large net numbers of Quanta is small.* This may be another approach to the concept of statistical equilibrium, an approach which stems directly from the introduction of the operator $N(\sigma)$ which appears naturally from our manner for obtaining operator identities valid in any representation. The demand, however, that our field equations be obtained in a manner independent of the representation has as a consequence altered the usual form for the generalized Schroedinger equation through the appearance of $N(\sigma)$ in (4.21).

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* Assuming that the matrix elements of the interaction do not involve powers of N' higher than unity.

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